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Bivariate normal-power series class of distributions: model, properties and applications

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Recently Mahmoudi and Mahmoodian (2017a) introduced a new class of distributions which obtained by compounding normal and power series class of distributions. This class of distributions are very flexible and can be used quite effectively to analysis skewed data. In this paper we proposed a new bivariate class of distributions with the normal-power series distributions marginals. Different properties of this new bivariate class of distributions have been studied. Bivariate normal power series class of distributions has five unknown parameters. The EM algorithm is used to determine the maximum likelihood estimates of the parameters. We illustrate the usefulness of the new class of distributions by means of an application to a real data set.

keywords: Normal distribution; Power series distributions; EM algorithm; Maximum likelihood estimation; Copula.

1 Introduction

The normal distribution is probably the most well-known statistical distribution and widely used to model many phenomena. Notice that normal distributions is symmetric. Many different fields of science such as engineering, economics, actuarial sciences and medicine, used asymmetry and skew data that are outside of the range allowed by the normal distribution, so it is necessary to introduce another model that can take into account these issues. Due to this reason, Azzalini (1985) discussed formally and popularized the univariate skew-normal distribution. A random variable Z is said to

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have a skew-normal distribution with parameter $\lambda \in \mathbb{R}$, if its probability density function (pdf) is given by

$$\phi(z;\lambda) = 2\phi(z)\Phi(\lambda z), \quad z \in \mathbb{R},$$

where $\phi(z)$ and $\Phi(z)$ are the standard normal density and cumulative distribution functions (cdf), respectively. This distribution and its variations have been discussed by several authors including Azzalini (1986), Henze (1986), Branco and Dey (2001), Loperfido (2001), Azzalini and Chiogna (2004), Arellano-Valle et al. (2006) and Sharafi and Behboodian (2008). Recently, Mahmoudi and Mahmoodian (2017a) by compounding normal and power series class of distributions introduced an alternative skewed model and named it normal-power series (NPS) class of distributions. They obtained several properties of the NPS distributions such as moments, maximum likelihood estimation procedure via an EM-algorithm and inference for a large sample.

Recently, Kundu and Gupta (2014) introduced a new bivariate distribution by compounding a bivariate Weibull distribution with a geometric distribution. The bivariate generalized exponential-power series, the bivariate generalized linear failure rate-power series, the bivariate Weibull-power series and the bivariate normal-geometric distributions introduced and studied by Jafari et al. (2018), Roozgar and Jafari (2015), Roozegar and Nadarajah (2016) and Mahmoudi and Mahmoodian (2017b).

In this paper, we introduce bivariate normal-power series (BNPS) class of distributions with the normal-power series distributions marginals. The BNPS class of distributions contain several lifetime models such as: bivariate normal-geometric (BNG), bivariate normal-Poisson (BNP), bivariate normal-logarithmic (BNL) and bivariate normal-binomial (BNB). Many properties of the joint distribution of order statistics can be used in establishing different properties of the proposed bivariate normal power series distributions. We provide the joint and conditional density functions, the joint cumulative and survival functions. It is observed that the generation of random samples from the proposed bivariate model is straightforward, hence simulation experiments can be performed quite conveniently. The proposed bivariate class of distributions have five parameters. We use an EM algorithm to estimate the model parameters. Moreover, it has a physical interpretation also.

The main aim of this paper is to introduce a bivariate distribution with continuous marginals and having a non-singular component which can be used to analyze data with negative, negative-positive and positive values with no ties in data. The main advantage of the proposed bivariate distribution is that it can have marginals with heavy tails. The proposed model has some interesting physical interpretations also. Hence, it may be more flexible than the existing models and it will give the practitioner one more option to choose a model among the possible class of bivariate models for analyzing negative and negative-positive data.

To begin with, we shall use the following notation throughout this paper: $\phi(\cdot)$ for the univariate standard normal pdf, $\phi_n(\cdot\,;\boldsymbol{\mu},\boldsymbol{\Sigma})$ for the pdf of $N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$ (n -variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$), $\Phi_n(\cdot\,;\boldsymbol{\mu},\boldsymbol{\Sigma})$ for the cdf of $N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$ (in both singular and non-singular cases), simply $\Phi_n(\cdot\,;\boldsymbol{\Sigma})$ for the case when

$\boldsymbol{\mu} = \mathbf{0}$. Furthermore, for $r, k \in \mathbb{N}$, let $\mathbf{1}_r$, $\mathbf{I}_r$ and $\mathbf{0}_{r\times k}$ denote the vector of ones, the identity matrix of dimension r, and $r \times k$ zero matrix, respectively.

The rest of the paper is organized as follows. In Section 2, we provide a brief review of the univariate normal power series class of distributions. The bivariate normal power series class of distributions are discussed in Section 3. In Section 4 different properties of the proposed bivariate normal-power series are given. Section 5 devotes with the copula form of these class of distributions. In Section 6, we present some special distributions which are studied in details. EM algorithm is presented in Section 7. Simulation study is given in Section 8. Applications to one real data set are given in Section 9. Finally, Section 10 concludes the paper.

2 Univariate normal-power series class of distributions

Let $X_1, .., X_N$ be a random sample from normal distribution with mean μ and variance σ^2 and N belongs to a power series distributions (truncated at zero) with the following probability mass function

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \tag{2.1}$$

where $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ and $\theta \in (0, s)$ (s can be ∞) is such that $C(\theta)$ is finite. Detailed properties of power series distributions can be found in Noack (1950). Here, $C'(\theta)$, $C''(\theta)$ and $C'''(\theta)$ denote the first, second and third derivatives of $C(\theta)$ with respect to θ, respectively. Moreover, N is independent of X_i's. If $X_{(N)} = \max(X_1, .., X_N)$, then the conditional cdf of $X_{(N)}|N = n$ is given by

$$G_{X_{(N)}|N=n}(x) = (\Phi(x; \mu, \sigma))^n,$$

where $\Phi(\cdot; \mu, \sigma)$ is cdf of normal distribution with mean μ and variance σ^2. The normal-power series class of distributions is defined by the marginal cdf of $X_{(N)}$:

$$F(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} (\Phi(x; \mu, \sigma)^n = \frac{C(\theta\Phi(x; \mu, \sigma))}{C(\theta)},\ x \in \mathbb{R}.$$

The corresponding pdf, survival and hazard rate functions are

$$\begin{aligned} f(x) &= \theta\phi(x; \mu, \sigma) \frac{C'(\theta\Phi(x; \mu, \sigma))}{C(\theta)}, \\ S(x) &= 1 - \frac{C(\theta\Phi(x; \mu, \sigma))}{C(\theta)}, \end{aligned}$$

and

$$h(x) = \theta\phi(x; \mu, \sigma) \frac{C'(\theta\phi(x; \mu, \sigma))}{C(\theta) - C(\theta\Phi(x; \mu, \sigma))}.$$

Hereafter, this distribution will be denoted by $NPS(\mu, \sigma, \theta)$. The moment generating function (mgf) and mean of $NPS(\mu, \sigma, \theta)$ can be obtained as

$$\begin{aligned} M_X(t) &= \exp\left(\frac{1}{2}t^2\right) \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \times n\Phi_{n-1}(\mathbf{1}_{n-1}t; \mathbf{I}_{n-1} + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T) \\ &= \exp\left(\frac{1}{2}t^2\right) E(N\Phi_{N-1}(\mathbf{1}_{N-1}t; \mathbf{I}_{N-1} + \mathbf{1}_{N-1}\mathbf{1}_{N-1}^T)), \end{aligned}$$

and

$$E(X) = \mu + \sigma \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \times \frac{(n-1)\Phi_{n-2}\left(\mathbf{0}; \mathbf{I}_{n-2} + \frac{1}{2}\mathbf{1}_{n-2}\mathbf{1}_{n-2}^T\right)}{2\sqrt{\pi}\Phi_{n-1}\left(\mathbf{0}; \mathbf{I}_{n-1} + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T\right)}.$$

This class of distributions contain several sub-models such as normal-geometric (NG), normal-Poisson (NP), normal-logarithmic (NL) and normal-binomial (NB) distributions as special cases. Detailed properties of normal-power series class of distributions can be found in Mahmoudi and Mahmoodian (2017a).

3 The BNPS class of distributions

The bivariate normal-power series class of distributions can be construct as follows. Let $\{X_1, \cdots, X_N\}$ and $\{Y_1, \cdots, Y_N\}$ be two sequence of mutually independent and identically distributed (i.i.d.) from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Also N has a power series distribution (truncated at zero) with the probability mass function given in (2.1) and is independent of X_i's and Y_i's. Let

$$U_1 = \max(X_1, \cdots, X_N) \quad \text{and} \quad U_2 = \max(Y_1, \cdots, Y_N).$$

The joint cdf of (U_1, U_2) is

$$\begin{aligned} F_{U_1,U_2}(u_1, u_2) =& P(U_1 \leq u_1, U_2 \leq u_2) \\ =& \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} \left[\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2)\right]^n \\ =& \frac{C(\theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2))}{C(\theta)}. \end{aligned} \tag{3.1}$$

The bivariate random vector (U_1, U_2) is said to have a bivariate normal power series distributions, denoted by $BNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, if (U_1, U_2) has the joint cdf (3.1). The following interpretation can be given for the BNPS class of distributions.
Let $X_1, \cdots, X_N$ and $Y_1, \cdots, Y_N$ be the failure times of the N components in two inter independent systems then $(U_1, U_2) = (\max(X_1, \cdots, X_N), \max(Y_1, \cdots, Y_N))$ will be the failure times of the two systems if the components in both systems work in parallel case. It is quite simple to generate samples from a BNPS distribution. We present the following simple algorithm to generate (U_1, U_2) from the $BNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$.

Step 1 Generate N from the power series distributions and call its observed value equal n.

Step 2 Generate $\{X_1, \cdots, X_n\}$ and $\{Y_1, \cdots, Y_n\}$ from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$.

Step 3 Obtain $U_1 = \max(X_1, \cdots, X_n)$ and $U_2 = \max(Y_1, \cdots, Y_n)$, independently.

Step 4 Replicate Steps 1-3, m times to obtain a random sample of size m of BNPS class of distributions.

The joint probability distribution and survival functions of (U_1, U_2) are given by

$$\begin{aligned} f_{U_1,U_2}(u_1,u_2) &= \frac{\theta\phi(u_1;\mu_1,\sigma_1)\phi(u_2;\mu_2,\sigma_2)}{C(\theta)}\Big[C^{'}(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)) \\ &+ \theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)C^{''}(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))\Big], \end{aligned}$$

and

$$\begin{aligned} S_{U_1,U_2}(u_1,u_2) &= 1+\frac{C(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))}{C(\theta)}-\frac{C\left(\theta\Phi(u_1;\mu_1,\sigma_1)\right)}{C(\theta)} \\ &- \frac{C\left(\theta\Phi(u_2;\mu_2,\sigma_2)\right)}{C(\theta)}, \end{aligned}$$

respectively.

Proposition 1. *As* $\theta \to 0^+$ *we have*

$$\begin{aligned} \lim_{\theta\to 0^+} F_{U_1,U_2}(u_1,u_2) &= \lim_{\theta\to 0^+}\frac{C(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))}{C(\theta)} \\ &= \lim_{\theta\to 0^+}\frac{\sum_{n=1}^{\infty} a_n\theta^n\left[\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right]^n}{\sum_{n=1}^{\infty} a_n\theta^n} \\ &= \lim_{\theta\to 0^+}\frac{\sum_{n=1}^{c-1} a_n\theta^n\left[\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right]^n}{\sum_{n=1}^{c-1} a_n\theta^n + a_c\theta^c + \sum_{n=c+1}^{\infty} a_n\theta^n} \\ &+ \frac{a_c\theta^c\left[\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right]^c}{\sum_{n=1}^{c-1} a_n\theta^n + a_c\theta^c + \sum_{n=c+1}^{\infty} a_n\theta^n} \\ &+ \lim_{\theta\to 0^+}\frac{\sum_{n=c+1}^{\infty} a_n\theta^n\left[\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right]^n}{\sum_{n=1}^{c-1} a_n\theta^n + a_c\theta^c + \sum_{n=c+1}^{\infty} a_n\theta^n} \\ &= \left[\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right]^c, \end{aligned}$$

where $c = \min\{n \in \mathbb{N} \colon a_n > 0\}$.

The following theorem provides the marginal and conditional distributions of the $BNPS$ class of distributions.

Theorem 1. *If* $(U_1, U_2) \sim BNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, *then*
(i) $U_i \sim NPS(\mu_i, \sigma_i, \theta),\ i = 1, 2.$

(ii) The conditional pdf of U_1 *given* $U_2 = u_2$ *is*

$$\begin{aligned} f_{U_1|U_2}(u_1|u_2) &= \frac{\phi(u_1;\mu_1,\sigma_1)C'(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))}{C'(\theta\Phi(u_2;\mu_2,\sigma_2))} \\ &+ \frac{\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\phi(u_1;\mu_1,\sigma_1)C''(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))}{C'(\theta\Phi(u_2;\mu_2,\sigma_2))}. \end{aligned}$$

(iii)

$$P(U_1 \le u_1 \mid U_2 = u_2) = \frac{\Phi(u_1;\mu_1,\sigma_1)}{C'(\theta\Phi(u_2;\mu_2,\sigma_2))} \times C'(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)).$$

Proof. The proof of parts (i) and (ii) can be obtained in a routine matter. For part (iii), we can write

$$\begin{aligned} P(U_1 \le u_1 \mid U_2 = u_2) &= \frac{P(U_1 \le u_1, U_2 = u_2)}{P(U_2 = u_2)} \\ &= \sum_{n=1}^{\infty} P(U_1 \le u_1 \mid N = n, U_2 = u_2) \times P(N = n \mid U_2 = u_2) \\ &= \sum_{n=1}^{\infty} \frac{(\Phi(u_1;\mu_1,\sigma_1))^n\, na_n\theta^{n-1}\,(\Phi(u_2;\mu_2,\sigma_2))^{n-1}}{C'(\theta\Phi(u_2;\mu_2,\sigma_2))} \\ &= \frac{\Phi(u_1;\mu_1,\sigma_1)}{C'(\theta\Phi(u_2;\mu_2,\sigma_2))} \sum_{n=1}^{\infty} na_n\theta^{n-1}\,(\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))^{n-1} \\ &= \frac{\Phi(u_1;\mu_1,\sigma_1)}{C'(\theta\Phi(u_2;\mu_2,\sigma_2))} \times C'(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)). \end{aligned}$$

□

Remark 1. *If we consider* $V_1 = \min(X_1, ..., X_N)$ *and* $V_2 = \min(Y_1, ..., Y_N)$, *another class of bivariate distributions is obtained with the following joint cumulative survival function:*

$$\overline{F}_{V_1,V_2}(v_1, v_2) = P(V_1 > v_1, V_2 > v_2) = \frac{C\left(\theta(1-\Phi(u_1;\mu_1,\sigma_1))(1-\Phi(u_2;\mu_2,\sigma_2))\right)}{C(\theta)}.$$

4 Properties

To better motivate the results developed in this section, we first provide a brief definition of the multivariate unified skew-normal (SUN) distributions. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two random vectors of dimensions m and n, respectively, and

$$\begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} \sim N_{m+n}\left(\begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{\xi} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \boldsymbol{\Omega} \end{pmatrix} \right).$$

The n-dimensional random vector $\mathbf{Z}$ is said to have the SUN distribution with parameter $\boldsymbol{\alpha} = (\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$, where $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\eta} \in \mathbb{R}^m$ are location vectors, $\boldsymbol{\Omega} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{\Gamma} \in \mathbb{R}^{m\times m}$ are dispersion matrices, and $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times m}$ is a skewness/shape matrix, denoted by $\mathbf{Z} \sim SUN_{n,m}(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$ or simply by $\mathbf{Z} \sim SUN_{n,m}(\boldsymbol{\alpha})$, if

$$\mathbf{Z} \overset{d}{=} \mathbf{V_2} \mid (\mathbf{V_1} > 0).$$

The density function of $\mathbf{Z}$ is [see Arellano-Valle and Azzalini (2006), Arellano-Valle et al. (2006) and Arellano-Valle and Genton (2010)]

$$f_{SUN_{n,m}}(\mathbf{z}; \boldsymbol{\alpha}) = \frac{\phi_n(\mathbf{z}; \boldsymbol{\xi}, \boldsymbol{\Omega})\, \Phi_m\left(\boldsymbol{\eta} + \boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}(\mathbf{z}-\boldsymbol{\xi}); \boldsymbol{\Gamma} - \boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\right)}{\Phi_m(\boldsymbol{\eta}; \boldsymbol{\Gamma})}.$$

Furthermore, when $\mathbf{Z} \sim SUN_{n,m}(\boldsymbol{\alpha})$, the mgf of $\mathbf{Z}$ is available in an explicit form and is given by

$$M_{SUN_{n,m}}(\mathbf{s}; \boldsymbol{\alpha}) = \frac{\exp\left(\boldsymbol{\xi}^T\mathbf{s} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Omega}\mathbf{s}\right) \Phi_m\left(\boldsymbol{\eta} + \boldsymbol{\Lambda}^T\mathbf{s}; \boldsymbol{\Gamma}\right)}{\Phi_m(\boldsymbol{\eta}; \boldsymbol{\Gamma})}. \tag{4.1}$$

Now, let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors of dimensions n, and $(\mathbf{X}^T, \mathbf{Y}^T)$ having a multivariate normal distribution

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim N_{2n}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mathbf{1}_n\mu_1 \\ \mathbf{1}_n\mu_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2\mathbf{I}_n & \mathbf{0}_{n\times n} \\ & \sigma_2^2\mathbf{I}_n \end{pmatrix}.$$

If $X_{(n)} = \max(X_1, \cdots, X_n)$ and $Y_{(n)} = \max(Y_1, \cdots, Y_n)$ be the largest order statistics obtained from $\mathbf{X}$ and $\mathbf{Y}$ respectively, then the joint pdf of $(X_{(n)}, Y_{(n)})$ is given by Pourmousa and Jamalizadeh (2014).

$$f_{X_{(n)},Y_{(n)}}(u_1, u_2) = f_{SUN_{2,2n-2}}(u_1, u_2; \boldsymbol{\alpha}), \tag{4.2}$$

where $\boldsymbol{\alpha} = (\boldsymbol{\xi}, \mathbf{0}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$, with

$$\begin{aligned}
\boldsymbol{\xi} &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \ \Omega = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}, \\
\boldsymbol{\Gamma} &= \begin{pmatrix} \sigma_1^2\left(\mathbf{I}_{n-1} + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T\right) & \mathbf{0}_{n-1\times n-1} \\ & \sigma_2^2\left(\mathbf{I}_{n-1} + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T\right) \end{pmatrix}, \\
\Lambda &= \begin{pmatrix} \mathbf{1}_{n-1}\sigma_1^2 & \mathbf{0}_{n-1\times n-1} \\ & \mathbf{1}_{n-1}\sigma_2^2 \end{pmatrix}.
\end{aligned}$$

In the following proposition, we present the mixture representation of $f_{U_1,U_2}(u_1, u_2)$.

Proposition 2. *The densities of BNPS class of distributions can be written as*

$$f_{U_1,U_2}(u_1,u_2) = \sum_{n=1}^{\infty} P(N=n) f_{X_{(n)},Y_{(n)}}(u_1,u_2), \tag{4.3}$$

where $f_{X_{(n)},Y_{(n)}}(u_1,u_2)$ *is the joint density function of* $(X_{(n)}, Y_{(n)})$ *in* (4.2).

Proposition 3. *If the random vector* $(U_1,U_2) \sim BNPS(\mu_1,\mu_2,\sigma_1,\sigma_2,\theta)$*, then*
(i) The mgf of (U_1,U_2) *is given by (for* $\mathbf{s} \in \mathbb{R}^{2\times 2}$ *)*

$$M_{U_1,U_2}(\mathbf{s}) = \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)} \times \frac{\exp\left(\boldsymbol{\xi}^T\mathbf{s} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Omega}\mathbf{s}\right)\Phi_{2n-2}\left(\boldsymbol{\Lambda}^T\mathbf{s};\boldsymbol{\Gamma}\right)}{\Phi_{2n-2}\left(\mathbf{0};\boldsymbol{\Gamma}\right)}.$$

(ii) The product moment $E(U_1U_2)$ *is given by*

$$\begin{aligned} E(U_1U_2) &= \\ &= \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)}\left[\mu_1 + \frac{n(n-1)\sigma_1}{2\sqrt{\pi}}\Phi_{n-2}\left(\mathbf{0};\mathbf{I}_{n-2}+\frac{1}{2}\mathbf{1}_{n-2}\mathbf{1}_{n-2}^T\right)\right] \\ &\times \left[\mu_2 + \frac{n(n-1)\sigma_2}{2\sqrt{\pi}}\Phi_{n-2}\left(\mathbf{0};\mathbf{I}_{n-2}+\frac{1}{2}\mathbf{1}_{n-2}\mathbf{1}_{n-2}^T\right)\right]. \end{aligned}$$

Proof. (i) we can write

$$\begin{aligned} M_{U_1,U_2}(\mathbf{s}) &= \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)} M_{X_{(n)},Y_{(n)}}(\mathbf{s}) \\ &= \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)} M_{SUN_{2,2n-2}}(\mathbf{s};\boldsymbol{\alpha}) \\ &= \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)} \times \frac{\exp\left(\boldsymbol{\xi}^T\mathbf{s} + \frac{1}{2}\mathbf{s}^T\boldsymbol{\Omega}\mathbf{s}\right)\Phi_{2n-2}\left(\boldsymbol{\Lambda}^T\mathbf{s};\boldsymbol{\Gamma}\right)}{\Phi_{2n-2}\left(\mathbf{0};\boldsymbol{\Gamma}\right)}. \end{aligned}$$

The proof of (ii) can be obtained as follows:

$$\begin{aligned} E(U_1U_2) &= E(E(X_{(N)}Y_{(N)} \mid N=n)) = \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)} E(X_{(n)})E(Y_{(n)}) \\ &= \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)}\left[\mu_1 + \frac{n(n-1)\sigma_1}{2\sqrt{\pi}}\Phi_{n-2}\left(\mathbf{0};\mathbf{I}_{n-2}+\frac{1}{2}\mathbf{1}_{n-2}\mathbf{1}_{n-2}^T\right)\right] \\ &\quad \times \left[\mu_2 + \frac{n(n-1)\sigma_2}{2\sqrt{\pi}}\Phi_{n-2}\left(\mathbf{0};\mathbf{I}_{n-2}+\frac{1}{2}\mathbf{1}_{n-2}\mathbf{1}_{n-2}^T\right)\right]. \end{aligned}$$

□

The stress-strength parameter, $R = P(U_1 < U_2)$, is useful for data analysis purposes. The following result gives the stress-strength parameter of BNPS models.

Proposition 4. *If* $(U_1, U_2) \sim BNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, *then*

$$R = P(U_1 < U_2) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} F_{SUN_{1,2n-2}}\left(\mathbf{0}; \boldsymbol{\alpha}^*\right),$$

where $F_{SUN_{1,2n-2}}\left(\cdot; \boldsymbol{\alpha}^*\right)$ *is the cdf of the univariate* $SUN_{1,2n-2}(\boldsymbol{\alpha}^*)$ *distribution, and* $\boldsymbol{\alpha}^* = (\mu_1 - \mu_2, \mathbf{0}, \sigma_1^2 + \sigma_2^2, \boldsymbol{\Gamma}, \boldsymbol{\Lambda})$, *with*

$$\begin{aligned}
\boldsymbol{\Gamma} &= \begin{pmatrix} \sigma_1^2\left(\mathbf{I}_{n-1} + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T\right) & \mathbf{0}_{n-1\times n-1} \\ & \sigma_2^2\left(\mathbf{I}_{n-1} + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T\right) \end{pmatrix}, \\
\Lambda &= \begin{pmatrix} \mathbf{1}_{n-1}\sigma_1^2 & \mathbf{0}_{n-1\times n-1} \\ & \mathbf{1}_{n-1}\sigma_2^2 \end{pmatrix}.
\end{aligned}$$

Proof. We have

$$P(U_1 < U_2) = P(X_{(N)} < Y_{(N)}) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} P(X_{(n)} < Y_{(n)}).$$

Now, we compute $P(X_{(n)} < Y_{(n)})$. For this purpose, let $\mathbf{X}$ and $\mathbf{Y}$ be partitioned as

$$\mathbf{X} = \begin{pmatrix} X_i \\ \mathbf{X}_{-i} \end{pmatrix}, \ \mathbf{Y} = \begin{pmatrix} Y_j \\ \mathbf{Y}_{-j} \end{pmatrix}.$$

We then have

$$\begin{aligned}
P(X_{(n)} < Y_{(n)}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} P(\mathbf{1}_{n-1}X_i - \mathbf{X}_{-i} > \mathbf{0}, \mathbf{1}_{n-1}Y_j - \mathbf{Y}_{-j} > \mathbf{0}\ ) \times \\
P(X_i - Y_j &\leq 0 \mid \mathbf{1}_{n-1}X_i - \mathbf{X}_{-i} > \mathbf{0}, \mathbf{1}_{n-1}Y_j - \mathbf{Y}_{-j} > \mathbf{0}),
\end{aligned}$$

Since, for $i = 1, \cdots, n$ and $j = 1, \cdots, n$

$$\begin{pmatrix} \mathbf{1}_{n-1}X_i - \mathbf{X}_{-i} \\ \mathbf{1}_{n-1}Y_j - \mathbf{Y}_{-j} \\ X_i - Y_j \end{pmatrix} \sim N_{2n}\left(\begin{pmatrix} \mathbf{0} \\ \mu_1 - \mu_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda} \\ & \sigma_1^2 + \sigma_2^2 \end{pmatrix}\right),$$

by using the definition of the univariate SUN distribution, we have

$$P(X_i - Y_j \leq 0 \mid \mathbf{1}_{n-1}X_i - \mathbf{X}_{-i} > \mathbf{0}, \mathbf{1}_{n-1}Y_j - \mathbf{Y}_{-j} > \mathbf{0}) = F_{SUN_{1,2n-2}}\left(\mathbf{0}; \boldsymbol{\alpha}^*\right),$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P(\mathbf{1}_{n-1}X_i - \mathbf{X}_{-i} > \mathbf{0}, \mathbf{1}_{n-1}Y_j - \mathbf{Y}_{-j} > \mathbf{0}) = 1,$$

which completes the proof. □

Proposition 5. *If* $(U_1, U_2) \sim BNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, *then*
(i) The cdf and pdf of $\max(U_1, U_2)$ *is*

$$F_{\max(U_1,U_2)}(u) = P(U_1 \leq u, U_2 \leq u) = \frac{C(\theta\Phi(u; \mu_1, \sigma_1)\Phi(u; \mu_2, \sigma_2))}{C(\theta)},$$

and

$$\begin{aligned} f_{\max(U_1,U_2)}(u) &= \frac{[\theta\phi(u; \mu_1, \sigma_1)\Phi(u; \mu_2, \sigma_2) + \theta\phi(u; \mu_2, \sigma_2)\Phi(u; \mu_1, \sigma_1)]}{C(\theta)} \\ &\times \; C^{'}(\theta\Phi(u; \mu_1, \sigma_1)\Phi(u; \mu_2, \sigma_2)). \end{aligned}$$

(ii) The cdf and pdf of $\min(U_1, U_2)$ *is given by*

$$\begin{aligned} F_{\min(U_1,U_2)}(u) &= P(\min(U_1, U_2) \leq u) = \frac{C(\theta\Phi\,(u; \mu_1, \sigma_1))}{C(\theta)} \\ &+ \; \frac{C(\theta\Phi\,(u; \mu_2, \sigma_2))}{C(\theta)} - \frac{C(\theta\Phi(u; \mu_1, \sigma_1)\Phi(u; \mu_2, \sigma_2))}{C(\theta)}, \end{aligned}$$

and

$$\begin{aligned} f_{\min(U_1,U_2)}(u) &= \frac{\theta\phi\,(u; \mu_1, \sigma_1)\, C^{'}(\theta\Phi\,(u; \mu_1, \sigma_1))}{C(\theta)} + \frac{\theta\phi\,(u; \mu_1, \sigma_1)\, C^{'}(\theta\Phi\,(u; \mu_2, \sigma_2))}{C(\theta)} \\ &- \; f_{\max(U_1,U_2)}(u). \end{aligned}$$

5 Special cases of BNPS class of distributions

In this section four important sub-models of BNPS class of distributions are studied in details. These models are bivariate normal-geometric (BNG), bivariate normal-Poisson (BNP), bivariate normal-logarithmic (BNL) and bivariate normal-binomial (BNB) distributions.

5.1 Bivariate normal-geometric distribution

In the geometric case, i.e., when $a_n = 1$ and $C(\theta) = \frac{\theta}{1-\theta}$ $(0 < \theta < 1)$, we obtain bivariate normal-geometric distribution, denoned by $BNG(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, with cdf

$$F_{U_1,U_2}(u_1, u_2) = \frac{(1-\theta)\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2)}{1 - \theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2)}.$$

The probability density and survival functions are

$$\begin{aligned} f_{U_1,U_2}(u_1, u_2) &= \frac{(1-\theta)\phi(u_1; \mu_1, \sigma_1)\phi(u_2; \mu_2, \sigma_2)}{(1 - \theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2))^2} \\ &\times \; \left[1 + \frac{2\theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2)}{(1 - \theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2))}\right], \end{aligned}$$

and

$$\begin{aligned} S_{U_1,U_2}(u_1,u_2) &= 1-\frac{(1-\theta)\Phi(u_1;\mu_1,\sigma_1)}{1-\theta\Phi(u_1;\mu_1,\sigma_1)}-\frac{(1-\theta)\Phi(u_2;\mu_2,\sigma_2)}{1-\theta\Phi(u_2;\mu_2,\sigma_2)} \\ &+ \frac{(1-\theta)\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)}{1-\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)}. \end{aligned}$$

respectively.

If $(U_1,U_2)\sim BNG(\mu_1,\mu_2,\sigma_1,\sigma_2,\theta)$, then mgf of (U_1,U_2) is given by

$$\begin{aligned} M_{U_1,U_2}(\mathbf{s}) &= \sum_{n=1}^{\infty}(1-\theta)\theta^{n-1}M_{SUN_{2,n-2}}(\mathbf{s};\boldsymbol{\theta}) \\ &= \sum_{n=1}^{\infty}(1-\theta)\theta^{n-1}\times\frac{\exp\left(\boldsymbol{\xi}^T\mathbf{s}+\frac{1}{2}\mathbf{s}^T\boldsymbol{\Omega}\mathbf{s}\right)\Phi_{2n-2}\left(\boldsymbol{\Lambda}^T\mathbf{s};\boldsymbol{\Gamma}\right)}{\Phi_{2n-2}\left(\mathbf{0};\boldsymbol{\Gamma}\right)}. \end{aligned}$$

Figures 1 and 2 show the BNG density function and contour plots for selected values θ where $\mu_1=\mu_2=0$ and $\sigma_1=\sigma_2=1$, respectively.
The contour plots can show the dependency between components (U_1,U_2) and their skewness. As seen in Figure 2 for $\theta=0.01$ (left top graph) the dependency between (U_1,U_2) is very poor and the graph shows the symmetry, as θ increases, the positive dependency is increased and the marginal distributions of U_1 and U_2 are left skew.

5.2 Bivariate normal-Poisson distribution

In the Poisson case, i.e, when $a_n=\frac{1}{n!}$ and $C(\theta)=e^\theta-1$ $(\theta>0)$, we obtain bivariate normal-Poisson distribution, denoned by $BNP(\mu_1,\mu_2,\sigma_1,\sigma_2,\theta)$, with cdf

$$F_{U_1,U_2}(u_1,u_2)=\frac{e^{\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)}-1}{e^\theta-1}.$$

The probability density and survival functions are

$$\begin{aligned} f_{U_1,U_2}(u_1,u_2) &= \frac{\theta\phi(u_1;\mu_1,\sigma_1)\phi(u_2;\mu_2,\sigma_2)\left(1+\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right)}{e^\theta-1} \\ &\times\ e^{\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)}, \end{aligned}$$

and

$$S_{U_1,U_2}(u_1,u_2)=1-\frac{e^{\theta\Phi(u_1;\mu_1,\sigma_1)}+e^{\theta\Phi(u_2;\mu_2,\sigma_2)}+e^{\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)}-1}{e^\theta-1},$$

respectively.

If $(U_1,U_2)\sim BNP(\mu_1,\mu_2,\sigma_1,\sigma_2,\theta)$, then the mgf of (U_1,U_2) is

$$\begin{aligned} M_{U_1,U_2}(\mathbf{s}) &= \sum_{n=1}^{\infty}\frac{\theta^n}{n!(e^\theta-1)}M_{SUN_{2,n-2}}(\mathbf{s};\boldsymbol{\theta}) \\ &= \sum_{n=1}^{\infty}\frac{\theta^n}{n!(e^\theta-1)}\times\frac{\exp\left(\boldsymbol{\xi}^T\mathbf{s}+\frac{1}{2}\mathbf{s}^T\boldsymbol{\Omega}\mathbf{s}\right)\Phi_{2n-2}\left(\boldsymbol{\Lambda}^T\mathbf{s};\boldsymbol{\Gamma}\right)}{\Phi_{2n-2}\left(\mathbf{0};\boldsymbol{\Gamma}\right)}. \end{aligned}$$

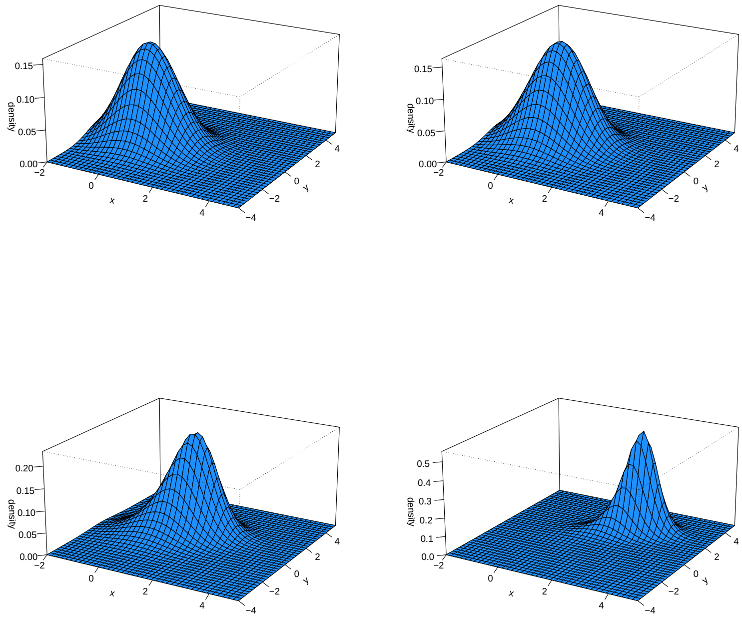

Figure 1: The pdf of BNG distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.01$ (left top), $\theta = 0.3$ (right top), $\theta = 0.8$ (left bottom), $\theta = 0.99$ (right bottom).

Figures 3 and 4 show the BNP density function and contour plots for selected values θ where $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$.
As seen in Figure 4 for $\theta = 0.01$ (left top graph) the dependency between (U_1, U_2) is very poor and the graph shows the symmetry, as θ increases (from $\theta = 2$ till $\theta = 8$), the positive dependency is increased and the marginal distributions of U_1 and U_2 are left skew. In BNP distribution the dependency between components is weaker than BNG distribution.

5.3 Bivariate normal-binomial distribution

In the binomial case, i.e, when $a_n = \begin{pmatrix} m \\ n \end{pmatrix}$ and $C(\theta) = (\theta + 1)^m - 1$ $(\theta > 0)$, where m $(n \leq m)$ is the number of replicas, we obtain bivariate normal-binomial distribution, denoned by $BNB(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, with cdf

$$F_{U_1,U_2}(u_1, u_2) = \frac{(\theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2) + 1)^m - 1}{(\theta + 1)^m - 1}.$$

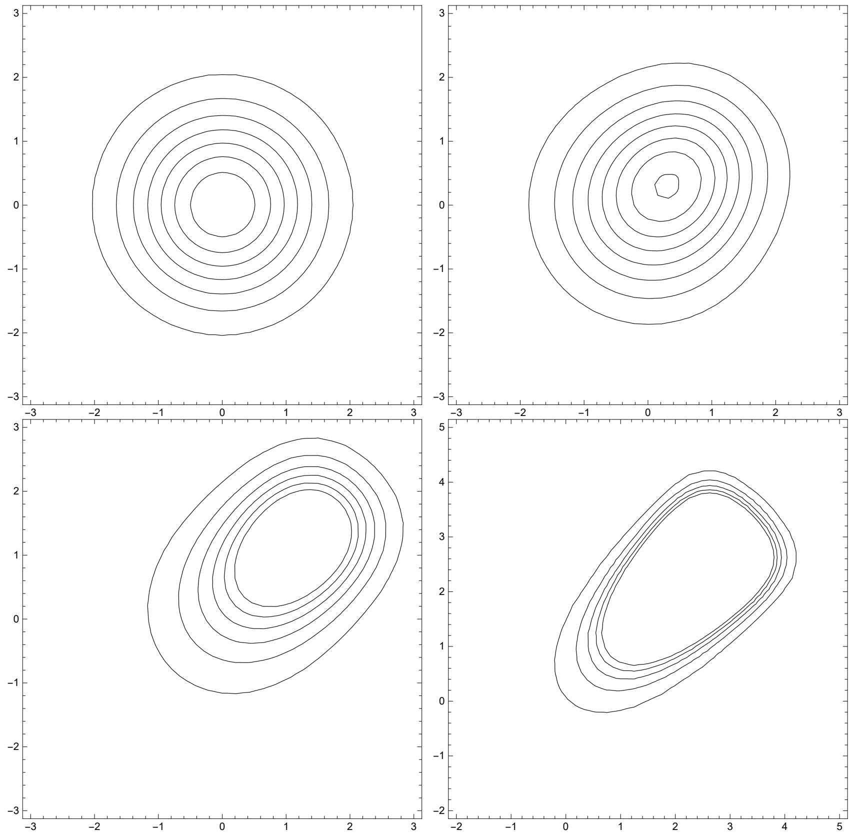

Figure 2: The contour plots of BNG distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.01$ (left top), $\theta = 0.3$ (right top), $\theta = 0.8$ (left bottom), $\theta = 0.99$ (right bottom).

The probability density and survival functions are given by

$$f_{U_1,U_2}(u_1,u_2) = \frac{\theta m \phi(u_1;\mu_1,\sigma_1)\phi(u_2;\mu_2,\sigma_2)(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)+1)^{m-1}}{((1+\theta)^m-1)} \times \left[1+\frac{(m-1)\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)}{(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)+1)}\right],$$

and

$$S_{U_1,U_2}(u_1,u_2) = 1-\frac{(\theta\Phi(u_1;\mu_1,\sigma_1)+1)^m}{(\theta+1)^m-1}-\frac{(\theta\Phi(u_2;\mu_2,\sigma_2)+1)^m}{(\theta+1)^m-1} + \frac{(\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)+1)^m-1}{(\theta+1)^m-1}.$$

respectively.

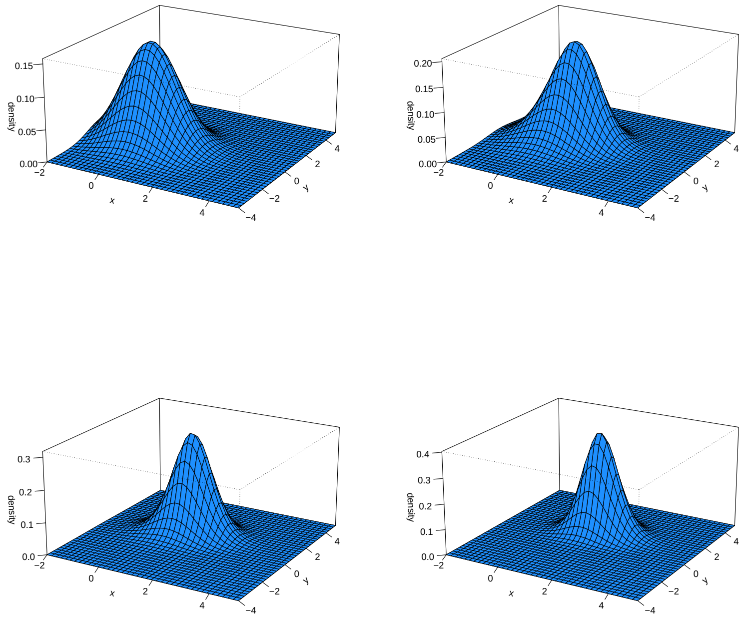

Figure 3: The pdf of BNP distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.01$ (left top), $\theta = 2$ (right top), $\theta = 5$ (left bottom), $\theta = 8$ (right bottom).

If $(U_1, U_2) \sim BNB(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, then the mgf of (U_1, U_2) is

$$\begin{aligned} M_{U_1,U_2}(\mathbf{s}) &= \sum_{n=1}^{\infty} \left(\begin{array}{c} m \\ n \end{array} \right) \frac{\theta^n}{(\theta+1)^m - 1} M_{SUN_{2,n-2}}(\mathbf{s}; \boldsymbol{\theta}) \\ &= \sum_{n=1}^{\infty} \left(\begin{array}{c} m \\ n \end{array} \right) \frac{\theta^n}{(\theta+1)^m - 1} \times \frac{\exp\left(\boldsymbol{\xi}^T \mathbf{s} + \frac{1}{2} \mathbf{s}^T \boldsymbol{\Omega} \mathbf{s}\right) \Phi_{2n-2}\left(\boldsymbol{\Lambda}^T \mathbf{s}; \boldsymbol{\Gamma}\right)}{\Phi_{2n-2}\left(\mathbf{0}; \boldsymbol{\Gamma}\right)}. \end{aligned}$$

Figures 5 and 6 show the BNB density function and contour plots for selected values θ where $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$ and $m = 5$.
As seen in Figure 6 for $\theta = 0.1$ (left top graph) the dependency between (U_1, U_2) is poor and the graph shows the symmetry (approximately), as θ increases (from $\theta = 2$ till $\theta = 8$), the positive dependency is increased and the marginal distributions of U_1 and U_2 are left skew. In BNB distribution the dependency between components is weaker than BNG distribution.

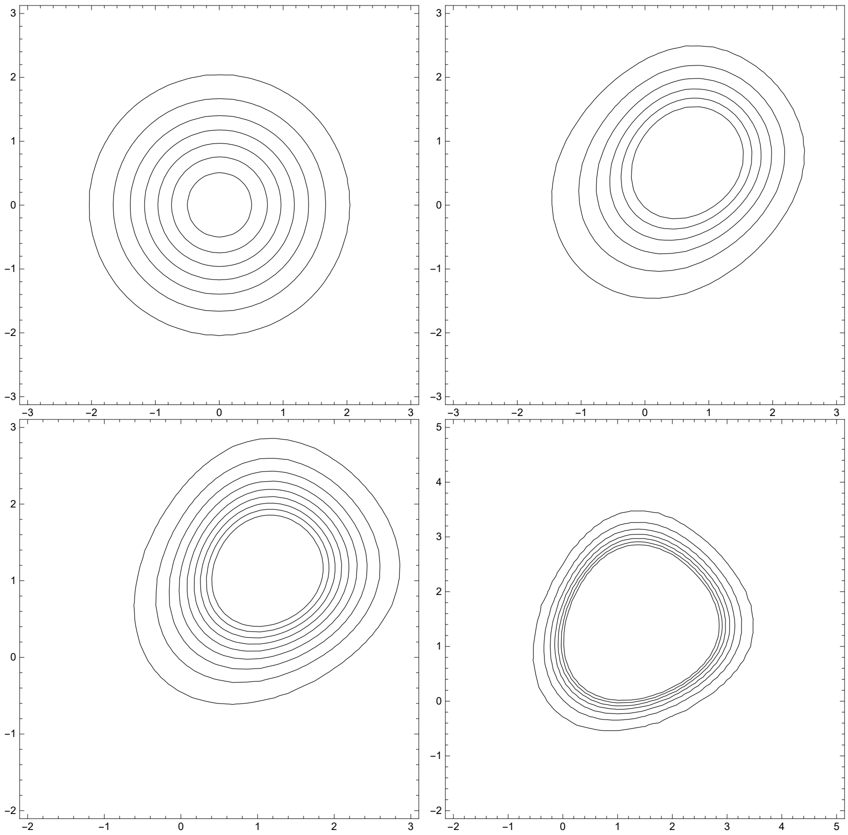

Figure 4: The contour plots of BNP distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.01$ (left top), $\theta = 2$ (right top), $\theta = 5$ (left bottom), $\theta = 8$ (right bottom).

5.4 Bivariate normal-logarithmic distribution

In the logarithmic case, i.e, when $a_n = \frac{1}{n}$ and $C(\theta) = -\log(1-\theta)$ $(0 < \theta < 1)$, we obtain bivariate normal logarithmic distribution, denoted by $BNL(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, with cdf

$$F_{U_1,U_2}(u_1, u_2) = \frac{\log(1 - \theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2))}{\log(1-\theta)}.$$

The probability density and survival functions are

$$\begin{aligned} f_{U_1,U_2}(u_1, u_2) &= -\frac{\theta\phi(u_1; \mu_1, \sigma_1)\phi(u_2; \mu_2, \sigma_2)}{\log(1-\theta)\,(1 - \theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2))} \\ &\times \left[1 + \frac{\theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2)}{(1 - \theta\Phi(u_1; \mu_1, \sigma_1)\Phi(u_2; \mu_2, \sigma_2))}\right], \end{aligned}$$

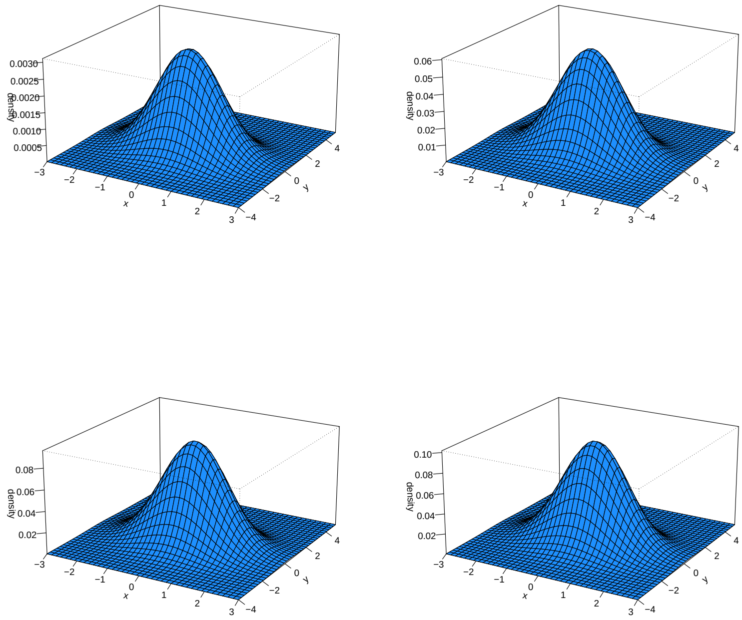

Figure 5: The pdf of BNB distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.1$ (left top), $\theta = 2$ (right top), $\theta = 5$ (left bottom), $\theta = 8$ (right bottom) .

and

$$\begin{aligned} S_{U_1,U_2}(u_1, u_2) &= 1 - \frac{\log(1-\theta\Phi(u_1;\mu_1,\sigma_1))}{\log(1-\theta)} - \frac{\log(1-\theta\Phi(u_2;\mu_2,\sigma_2))}{\log(1-\theta)} \\ &+ \frac{\log(1-\theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2))}{\log(1-\theta)}, \end{aligned}$$

respectively.

If $(U_1, U_2) \sim BNL(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, then the mgf of (U_1, U_2) is

$$\begin{aligned} M_{U_1,U_2}(\mathbf{s}) &= -\sum_{n=1}^{\infty} \frac{\theta^n}{n\log(1-\theta)} \times M_{SUN_{2,n-2}}(\mathbf{s};\boldsymbol{\theta}) \\ &= -\sum_{n=1}^{\infty} \frac{\theta^n}{n\log(1-\theta)} \times \frac{\exp\left(\boldsymbol{\xi}^T\mathbf{s}+\frac{1}{2}\mathbf{s}^T\boldsymbol{\Omega}\mathbf{s}\right)\Phi_{2n-2}\left(\boldsymbol{\Lambda}^T\mathbf{s};\boldsymbol{\Gamma}\right)}{\Phi_{2n-2}\left(\mathbf{0};\boldsymbol{\Gamma}\right)}. \end{aligned}$$

Figures 7 and 8 show the BNL density function and contour plots for selected values θ where $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$.

As seen in Figure 8 for $\theta = 0.01$ (left top graph) the dependency between (U_1, U_2) is

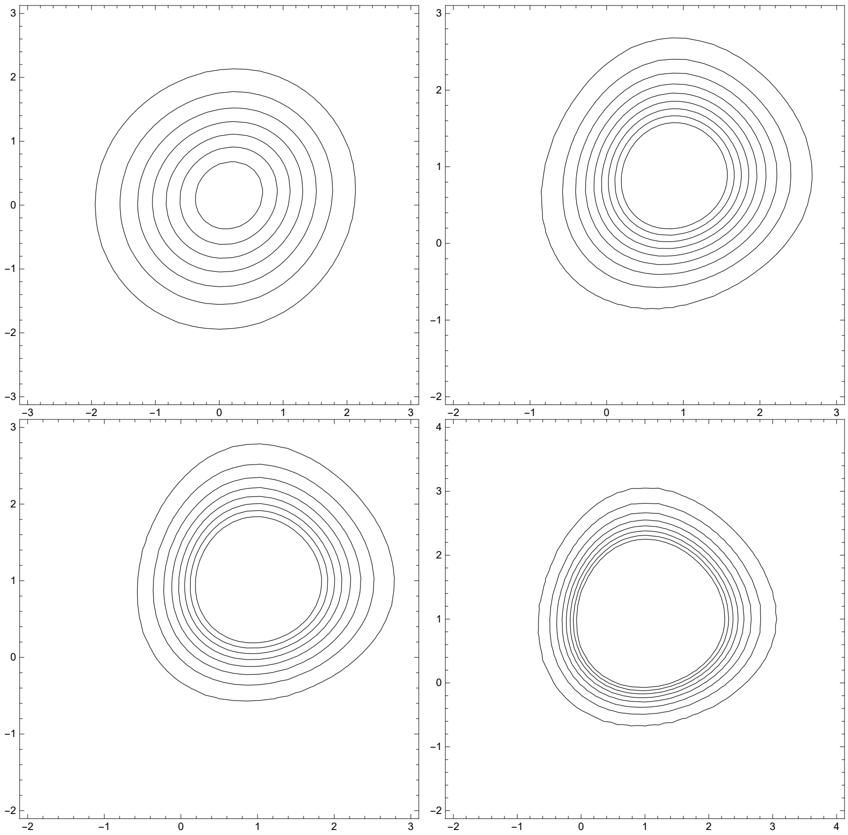

Figure 6: The contour plot of BNB distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.1$ (left top), $\theta = 2$ (right top), $\theta = 5$ (left bottom), $\theta = 8$ (right bottom) .

very poor and the graph shows the symmetry, as θ increases (from $\theta = 0.3$ till $\theta = 0.99$), the positive dependency is increased and the marginal distributions of U_1 and U_2 are left skew. In BNL distribution the dependency between components is stronger than BNP and BNB distributions.

6 Copula representation

Let $F_{X,Y}$ be a joint distribution function with continuous marginals F_X and F_Y. Then there exists a unique copula $A : [0,1] \times [0,1] \to [0,1]$ such that

$$F_{X,Y}(x,y) = A(F_X(x), F_Y(y)).$$

Moreover,

$$A(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)).$$

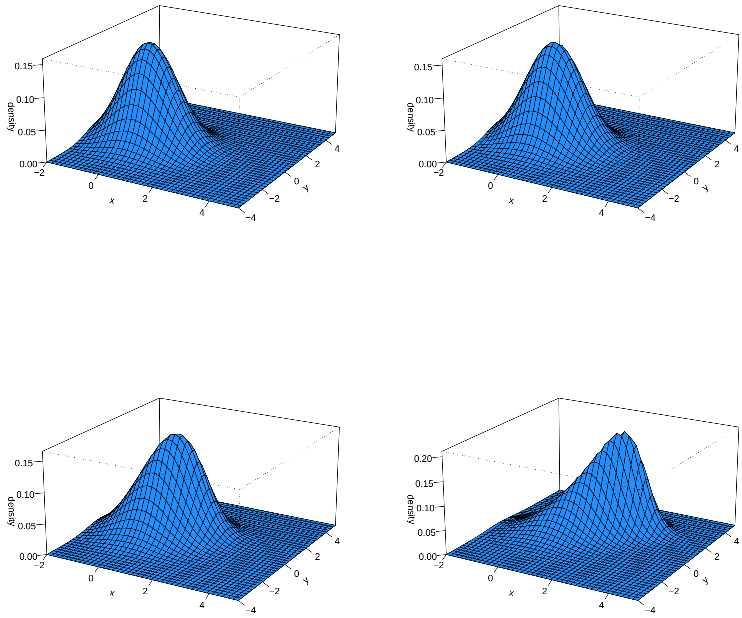

Figure 7: The pdf of BNL distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.01$ (left top), $\theta = 0.3$ (right top), $\theta = 0.8$ (left bottom), $\theta = 0.99$ (right bottom).

It can be shown by some calculation that if $(U_1, U_2) \sim BNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \theta)$, then the corresponding copulas $A(u, v)$ is

$$A(u,v) = \frac{C\left[\frac{1}{\theta} C^{-1}(uC(\theta)) C^{-1}(vC(\theta))\right]}{C(\theta)},$$

for all $u, v \in [0, 1]$. For example, in the geometric case, i.e, $C(\theta) = \frac{\theta}{1-\theta}$ $(0 < \theta < 1)$, we have

$$A(u,v) = \frac{uv}{(1 - \theta\,(1-u)\,(1-v))}.$$

This copula is a member of the Archimedean family of copulas with the strict generator $\varphi(t) = \log\left(\frac{1-\theta(1-t)}{t}\right)$ and it is known as the Ali-Mikhail-Haq copula (see Ali et al. , 1978).
In the Poisson case when $C(\theta) = e^{\theta} - 1 (\theta > 0)$, we have

$$A(u,v) = \frac{e^{\frac{1}{\theta}\log(1+u(e^{\theta}-1))\log(1+v(e^{\theta}-1))} - 1}{e^{\theta} - 1}.$$

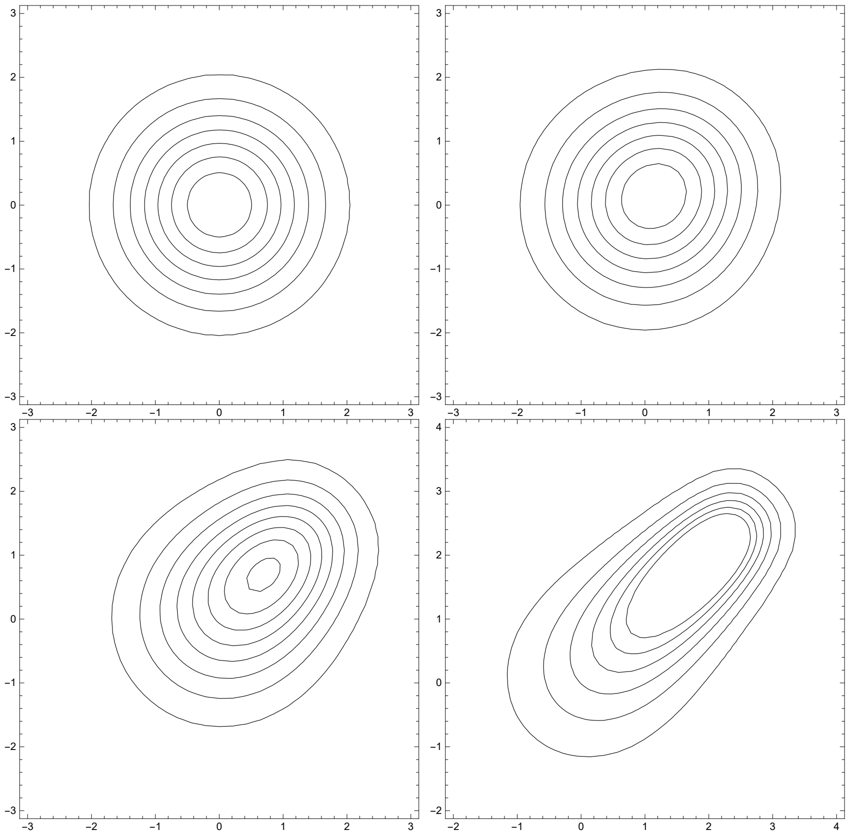

Figure 8: The contour plot of BNL distribution when $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\theta = 0.01$ (left top), $\theta = 0.3$ (right top), $\theta = 0.8$ (left bottom), $\theta = 0.99$ (right bottom) .

This copula is a member of the Archimedean family of copulas with the strict generator $\varphi(t) = -\log\left(\frac{\log\left(t\left(e^{\theta}-1\right)+1\right)}{\theta}\right)$.
In the binomial case when $C(\theta) = (\theta+1)^m - 1 \ \ (\theta > 0)$, we have

$$A(u,v) = \frac{\left\{\frac{1}{\theta}\left[\left(u\left((\theta+1)^m-1\right)+1\right)^{\frac{1}{m}}-1\right]\left[\left(v\left((\theta+1)^m-1\right)+1\right)^{\frac{1}{m}}-1\right]+1\right\}^m-1}{(\theta+1)^m-1}.$$

This copula is a member of the Archimedean family of copulas with the strict generator $\varphi(t) = -\log\left[\frac{[t((\theta+1)^m-1)+1]^{\frac{1}{m}}-1}{\theta}+1\right]$.
In the logarithmic case when $C(\theta) = -\log(1-\theta)$, $(0 < \theta < 1)$, we have

$$A(u,v) = -\frac{1}{\theta^*}\log\left(1+\frac{(e^{-\theta^* u}-1)(e^{-\theta^* v}-1)}{e^{-\theta^*}-1}\right),$$

where $\theta^* = -\log(1-\theta)$. This copula is a member of the Archimedean family of copulas with the strict generator $\varphi(t) = -\log\left(\frac{e^{-\theta^* t}}{e^{-\theta^*}-1}\right)$ and it is known as the Frank copula, see Frank (1979).

7 Inference

In this section, we consider estimation of unknown parameters of the BNPS distributions. Let $\{(u_{11}, u_{21}), \cdots, (u_{1n}, u_{2n})\}$ be a bivariate sample of size n from BNPS with parameters $\mathbf{\Psi} = (\mu_1, \sigma_1, \mu_2, \sigma_2, \theta)$. The log-likelihood function can be written as

$$\begin{aligned} l_n &= l_n(\mathbf{\Psi}) = n\log(\theta) - n\log(C(\theta)) - n\log(\sigma_1) - n\log(\sigma_1) - 2n\log(2\pi) \\ &- \frac{1}{2}\sum_{i=1}^{n} z_{1i}^2 - \frac{1}{2}\sum_{i=1}^{n} z_{2i}^2 + \sum_{i=1}^{n}\log\{C^{'}(\theta\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2) \\ &+ \theta\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2)C^{''}(\theta\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2))\}, \end{aligned} \tag{7.1}$$

where $z_{1i} = \frac{u_{1i}-\mu_1}{\sigma_1}$ and $z_{2i} = \frac{u_{2i}-\mu_2}{\sigma_2}$. The maximum likelihood estimators (MLEs) can be obtained by maximizing (7.1) with respect to the unknown parameters. Clearly, MLEs cannot be obtained in closed forms. We propose to use EM algorithm to compute the MLEs. The EM algorithm is a very powerful tool in handling the incomplete data problem (Dempster et al. , 1997; McLachlan and Krishnan , 1997). Let the complete-data be $(U_{11}, U_{21}), ..., (U_{1n}, U_{2n})$ with observed values $(u_{11}, u_{21}), ..., (u_{1n}, u_{2n})$ and the hypothetical random variable $Z_1, ..., Z_n$. We define a hypothetical complete-data distribution with a joint probability density function in the form

$$g(z, u_1, u_2; \mathbf{\Psi}) = \frac{a_z\theta^z}{C(\theta)} z^2\phi(u_1;\mu_1,\sigma_1)\phi(u_2;\mu_2,\sigma_2)\left[\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)\right]^{z-1},$$

where μ_1, $\mu_2 \in \mathbb{R}$, σ_1, $\sigma_2 > 0$ and $z \in \mathbb{N}$. Suppose $\mathbf{\Psi}^{(r)} = (\mu_1^{(r)}, \sigma_1^{(r)}, \mu_2^{(r)}, \sigma_2^{(r)}, \theta^{(r)})$ is the current estimate (in the rth iteration) of $\mathbf{\Psi}$. Then, the E-step of an EM cycle requires the expectation of $(Z|\, U_1, U_2;\ \mathbf{\Psi})$. Consider $\theta_* = \theta\Phi(u_1;\mu_1,\sigma_1)\Phi(u_2;\mu_2,\sigma_2)$, then the probability density function of Z given $U_1 = u_1, U_2 = u_2$ is given by

$$g(z|u_1, u_2) = \frac{a_z z^2\left[\theta_*\right]^{z-1}}{C'(\theta_*) + \theta_* C''(\theta_*)},$$

and its expected value is given by

$$E\left(Z|u_1, u_2; \mathbf{\Psi}\right) = \frac{\theta_*^2 C'''(\theta_*) + 3\theta_* C''(\theta_*) + C^{'}(\theta_*)}{C'(\theta_*) + \theta_* C''(\theta_*)}.$$

By using the maximum likelihood estimation over $\mathbf{\Psi}$, with the missing Z's replaced by their conditional expectations given above, the M-step of EM cycle is completed. The log-likelihood of the model parameters for the complete data set is

$$
\begin{aligned}
& l_n^*(\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}; \mu_1, \mu_2, \sigma_1, \sigma_2, \theta) \\
\propto \; & \sum_{i=1}^n z_i \log\theta - n\log\sigma_1 - n\log\sigma_2 - \frac{1}{2\sigma_1^2}\sum_{i=1}^n (u_{1i}-\mu_1)^2 - \frac{1}{2\sigma_2^2}\sum_{i=1}^n (u_{2i}-\mu_2)^2 \\
& + \sum_{i=1}^n (z_i-1)\log\Phi(u_{1i};\mu_1,\sigma_1) + \sum_{i=1}^n (z_i-1)\log\Phi(u_{2i};\mu_2,\sigma_2) - n\log(C(\theta)).
\end{aligned}
$$

and the components of the score vector, $U_C\left(\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}; \mathbf{\Psi}\right)$, are

$$
\begin{aligned}
\frac{\partial l_n^*}{\partial \mu_1} &= \frac{1}{\sigma_1^2}\sum_{i=1}^n (u_{1i}-\mu_1) - \sum_{i=1}^n (z_i-1)\,\frac{\phi(u_{1i};\mu_1,\sigma_1)}{\Phi(u_{1i};\mu_1,\sigma_1)}, \\
\frac{\partial l_n^*}{\partial \mu_2} &= \frac{1}{\sigma_2^2}\sum_{i=1}^n (u_{2i}-\mu_2) - \sum_{i=1}^n (z_i-1)\,\frac{\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{2i};\mu_2,\sigma_2)}, \\
\frac{\partial l_n^*}{\partial \sigma_1} &= -\frac{n}{\sigma_1} + \frac{1}{\sigma_1^3}\sum_{i=1}^n (u_{1i}-\mu_1)^2 - \frac{1}{\sigma_1}\sum_{i=1}^n (z_i-1)\,\frac{(u_{1i}-\mu_1)\,\phi(u_{1i};\mu_1,\sigma_1)}{\Phi(u_{1i};\mu_1,\sigma_1)}, \\
\frac{\partial l_n^*}{\partial \sigma_2} &= -\frac{n}{\sigma_2} + \frac{1}{\sigma_2^3}\sum_{i=1}^n (u_{2i}-\mu_2)^2 - \frac{1}{\sigma_2}\sum_{i=1}^n (z_i-1)\,\frac{(u_{2i}-\mu_2)\,\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{2i};\mu_2,\sigma_2)}, \\
\frac{\partial l_n^*}{\partial \theta} &= \frac{1}{\theta}\sum_{i=1}^n z_i - n\frac{C'(\theta)}{C(\theta)}.
\end{aligned}
$$

The maximum likelihood estimates can be obtained from the iterative algorithm given by

$$
\begin{aligned}
\frac{1}{\left(\widehat{\sigma}_1^{(k)}\right)^2}\sum_{i=1}^n \left(u_{1i} - \widehat{\mu}_1^{(k+1)}\right) - \sum_{i=1}^n \left(\widehat{z}_i^{(k)} - 1\right)\frac{\phi(u_{1i};\widehat{\mu}_1^{(k+1)},\widehat{\sigma}_1^{(k)})}{\Phi(u_{1i};\widehat{\mu}_1^{(k+1)},\widehat{\sigma}_1^{(k)})} &= 0, \\
\frac{1}{\left(\widehat{\sigma}_2^{(k)}\right)^2}\sum_{i=1}^n \left(u_{2i} - \widehat{\mu}_2^{(k+1)}\right) - \sum_{i=1}^n \left(\widehat{z}_i^{(k)} - 1\right)\frac{\phi(u_{2i};\widehat{\mu}_2^{(k+1)},\widehat{\sigma}_2^{(k)})}{\Phi(u_{2i};\widehat{\mu}_2^{(k+1)},\widehat{\sigma}_2^{(k)})} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{n}{\widehat{\sigma}_1^{(k+1)}} &- \frac{1}{\left(\widehat{\sigma}_1^{(k+1)}\right)^3}\sum_{i=1}^{n}\left(u_{1i}-\widehat{\mu}_1^{(k)}\right)^2 \\
&+ \frac{1}{\widehat{\sigma}_1^{(k+1)}}\sum_{i=1}^{n}\left(\widehat{z}_i^{(k)}-1\right)\frac{\left(u_{1i}-\widehat{\mu}_1^{(k)}\right)\phi(u_{1i};\widehat{\mu}_1^{(k)},\widehat{\sigma}_1^{(k+1)})}{\Phi(u_{1i};\widehat{\mu}_1^{(k)},\widehat{\sigma}_1^{(k+1)})} = 0, \\
\frac{n}{\widehat{\sigma}_2^{(k+1)}} &- \frac{1}{\left(\widehat{\sigma}_2^{(k+1)}\right)^3}\sum_{i=1}^{n}\left(u_{2i}-\widehat{\mu}_2^{(k)}\right)^2 \\
&+ \frac{1}{\widehat{\sigma}_2^{(k+1)}}\sum_{i=1}^{n}\left(\widehat{z}_i^{(k)}-1\right)\frac{\left(u_{2i}-\widehat{\mu}_2^{(k)}\right)\phi(u_{2i};\widehat{\mu}_2^{(k)},\widehat{\sigma}_2^{(k+1)})}{\Phi(u_{2i};\widehat{\mu}_2^{(k)},\widehat{\sigma}_2^{(k+1)})} = 0,
\end{aligned}
$$

$$
\widehat{\theta}^{(k+1)} = \frac{C(\widehat{\theta}^{(k+1)})}{nC'(\widehat{\theta}^{(k+1)})}\sum_{i=1}^{n}\widehat{z}_i^{(k)},
$$

where $\widehat{\mu}_1^{(k)}$, $\widehat{\mu}_2^{(k)}$, $\widehat{\sigma}_1^{(k)}$, $\widehat{\sigma}_2^{(k)}$ and $\widehat{\theta}^{(k)}$ are found numerically. Here, for $i=1,...,n$, we have that

$$
\widehat{z}_i^{(k)} = E\left(Z|U_1=u_{1i},U_2=u_{2i};\mu_1^{(k)},\mu_2^{(k)},\sigma_1^{(k)},\sigma_2^{(k)},\theta^{(k)}\right).
$$

In this part we obtain the standard errors of the estimators from the EM-algorithm by using the results of Louis (1982). The elements of the 5×5 observed information matrix $I_c\left(\mathbf{\Psi};\mathbf{u}_1,\mathbf{u}_2,\mathbf{z}\right) = -\left[\frac{\partial U_C(\mathbf{u}_1,\mathbf{u}_2,\mathbf{z};\mathbf{\Psi})}{\partial\mathbf{\Psi}}\right]$ are given by

$$
\begin{aligned}
\frac{\partial^2 l_n^*}{\partial\mu_1^2} &= \frac{n}{\sigma_1^2}+\sum_{i=1}^{n}(z_i-1)\frac{\left(\frac{u_{i1}-\mu_1}{\sigma_1^2}\right)\phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{1i};\mu_1,\sigma_1)+\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)}, \\
\frac{\partial^2 l_n^*}{\partial\mu_2^2} &= \frac{n}{\sigma_2^2}+\sum_{i=1}^{n}(z_i-1)\frac{\left(\frac{u_{i2}-\mu_2}{\sigma_2^2}\right)\phi(u_{2i};\mu_2,\sigma_2)\Phi(u_{1i};\mu_2,\sigma_2)+\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l_n^*}{\partial\mu_1\partial\sigma_1} &= \frac{\partial^2 l_n^*}{\partial\sigma_1\partial\mu_1} \\
&= \frac{2}{\sigma_1^3}\sum_{i=1}^{n}(u_{1i}-\mu_1)+\frac{1}{\sigma_1^2}\sum_{i=1}^{n}(z_i-1)\frac{(u_{1i}-\mu_1)\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)} \\
&+ \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(z_i-1)\frac{\left((\frac{u_{1i}-\mu_1}{\sigma_1})^2-1\right)\phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l_n^*}{\partial\mu_2\partial\sigma_2} &= \frac{\partial^2 l_n^*}{\partial\sigma_2\partial\mu_2} \\
&= \frac{2}{\sigma_2^3}\sum_{i=1}^{n}(u_{2i}-\mu_2) + \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{2i}-\mu_2)\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)} \\
&+ \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(z_i-1)\,\frac{\left((\frac{u_{2i}-\mu_2}{\sigma_2})^2-1\right)\phi(u_{2i};\mu_2,\sigma_2)\Phi(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l_n^*}{\partial\sigma_1^2} &= -\frac{n}{\sigma_1^2} - \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{1i}-\mu)\,\phi(u_{1i};\mu_1,\sigma_1)}{\Phi(u_{1i};\mu_1,\sigma_1)} \\
&+ \frac{3}{\sigma_1^4}\sum_{i=1}^{n}(u_{1i}-\mu_1)^2 + \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{1i}-\mu_1)^2\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)} \\
&+ \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{1i}-\mu_1)\left((\frac{u_{1i}-\mu_1}{\sigma_1})^2-1\right)\phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l_n^*}{\partial\sigma_2^2} &= -\frac{n}{\sigma_2^2} - \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{2i}-\mu)\,\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{2i};\mu_2,\sigma_2)} \\
&+ \frac{3}{\sigma_2^4}\sum_{i=1}^{n}(u_{2i}-\mu_2)^2 + \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{2i}-\mu_2)^2\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)} \\
&+ \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(z_i-1)\,\frac{(u_{2i}-\mu_2)\left((\frac{u_{2i}-\mu_2}{\sigma_2})^2-1\right)\phi(u_{2i};\mu_2,\sigma_2)\Phi(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l_n^*}{\partial\theta^2} &= \frac{1}{\theta^2}\sum_{i=1}^{n} z_i + n\frac{C''(\theta)C(\theta)-(C'(\theta))^2}{C^2(\theta)}, \\
\frac{\partial^2 l_n^*}{\partial\mu_1\partial\mu_2} &= \frac{\partial^2 l_n^*}{\partial\mu_2\partial\mu_1} = \frac{\partial^2 l_n^*}{\partial\mu_1\partial\sigma_2} = \frac{\partial^2 l_n^*}{\partial\sigma_2\partial\mu_1} = \frac{\partial^2 l_n^*}{\partial\mu_1\partial\theta} = \frac{\partial^2 l_n^*}{\partial\theta\partial\mu_1} = \frac{\partial^2 l_n^*}{\partial\mu_2\partial\sigma_1} \\
&= \frac{\partial^2 l_n^*}{\partial\sigma_1\partial\mu_2} = 0, \\
\frac{\partial^2 l_n^*}{\partial\sigma_1\partial\theta} &= \frac{\partial^2 l_n^*}{\partial\sigma_2\partial\theta} = \frac{\partial^2 l_n^*}{\partial\mu_2\partial\theta} = \frac{\partial^2 l_n^*}{\partial\theta\partial\mu_2} = \frac{\partial^2 l_n^*}{\partial\sigma_2\partial\sigma_1} = \frac{\partial^2 l_n^*}{\partial\sigma_2\partial\sigma_1} = \frac{\partial^2 l_n^*}{\partial\theta\partial\sigma_1} \\
&= \frac{\partial^2 l_n^*}{\partial\theta\partial\sigma_2} = 0.
\end{aligned}
$$

Taking the conditional expectation of $I_c(\boldsymbol{\Psi};\mathbf{u}_1,\mathbf{u}_2,\mathbf{z})$ given $(\mathbf{u}_1,\mathbf{u}_2)$, we obtain the 5×5 matrix

$$
l_c(\boldsymbol{\Psi};\mathbf{u}_1,\mathbf{u}_2) = E\left(I_c(\boldsymbol{\Psi};\mathbf{u}_1,\mathbf{u}_2,\mathbf{z}) \mid (\mathbf{u_1},\mathbf{u_2})\right) = [c_{ij}], \tag{7.2}
$$

where

$$
\begin{aligned}
c_{11} &= \frac{n}{\sigma_1^2} + \sum_{i=1}^{n} \frac{(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\left(\frac{u_{i1}-\mu_1}{\sigma_1^2}\right)\phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{1i};\mu_1,\sigma_1)+\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},\\
c_{22} &= \frac{n}{\sigma_2^2} + \sum_{i=1}^{n} (E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{\left(\frac{u_{i2}-\mu_2}{\sigma_2^2}\right)\phi(u_{2i};\mu_2,\sigma_2)\Phi(u_{1i};\mu_2,\sigma_2)+\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

$$
\begin{aligned}
c_{13} &= c_{31} = \frac{2}{\sigma_1^3}\sum_{i=1}^{n}(u_{1i}-\mu_1) + \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{1i}-\mu_1)\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)}\\
&+ \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{\left(\frac{u_{1i}-\mu_1}{\sigma_1})^2-1\right)\phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},
\end{aligned}
$$

$$
\begin{aligned}
c_{24} &= c_{42} = \frac{2}{\sigma_2^3}\sum_{i=1}^{n}(u_{2i}-\mu_2) + \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{2i}-\mu_2)\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)}\\
&+ \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{\left(\frac{u_{2i}-\mu_2}{\sigma_2})^2-1\right)\phi(u_{2i};\mu_2,\sigma_2)\Phi(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

$$
\begin{aligned}
c_{33} &= -\frac{n}{\sigma_1^2} - \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{1i}-\mu_1)\,\phi(u_{1i};\mu_1,\sigma_1)}{\Phi(u_{1i};\mu_1,\sigma_1)}\\
&+ \frac{3}{\sigma_1^4}\sum_{i=1}^{n}(u_{1i}-\mu_1)^2 + \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{1i}-\mu_1)^2\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)}\\
&+ \frac{1}{\sigma_1^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{1i}-\mu_1)\left((\frac{u_{1i}-\mu_1}{\sigma_1})^2-1\right)\phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},
\end{aligned}
$$

$$
\begin{aligned}
c_{44} &= -\frac{n}{\sigma_2^2} - \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{2i}-\mu_2)\,\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{2i};\mu_2,\sigma_2)}\\
&+ \frac{3}{\sigma_2^4}\sum_{i=1}^{n}(u_{2i}-\mu_2)^2 + \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{2i}-\mu_2)^2\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)}\\
&+ \frac{1}{\sigma_2^2}\sum_{i=1}^{n}(E(Z_i|\mathbf{u}_1,\mathbf{u}_2)-1)\frac{(u_{2i}-\mu_2)\left((\frac{u_{2i}-\mu_2}{\sigma_2})^2-1\right)\phi(u_{2i};\mu_2,\sigma_2)\Phi(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

$$
\begin{aligned}
c_{55} &= \frac{1}{\theta^2}\sum_{i=1}^{n}E(Z_i|\mathbf{u}_1,\mathbf{u}_2) + n\frac{C''(\theta)C(\theta)-(C'(\theta))^2}{C^2(\theta)},\\
c_{12} &= c_{21} = c_{15} = c_{51} = c_{14} = c_{41} = c_{23} = c_{32} = 0,\\
c_{25} &= c_{52} = c_{34} = c_{43} = c_{35} = c_{53} = c_{45} = c_{54} = 0,
\end{aligned}
$$

Moving now to the computation of $l_m(\mathbf{\Psi};\mathbf{u}_1,\mathbf{u}_2)$ as

$$l_m(\mathbf{\Psi};;\mathbf{u}_1,\mathbf{u}_2) = Var[U_C(\mathbf{u}_1,\mathbf{u}_2,\mathbf{z};\mathbf{\Psi}) \mid \mathbf{u}_1,\mathbf{u}_2] = [v_{ij}], \tag{7.3}$$

where

$$\begin{aligned}
v_{11} &= \sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},\\
v_{22} &= \sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},\\
v_{33} &= \frac{1}{\sigma_1^2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{1i}-\mu_1)^2\,\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},\\
v_{44} &= \frac{1}{\sigma_2^2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{2i}-\mu_2)^2\,\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},\\
v_{55} &= \frac{1}{\theta^2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2).
\end{aligned}$$

$$\begin{aligned}
v_{12} &= v_{21} = \sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{\phi(u_{1i};\mu_1,\sigma_1)\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2)},\\
v_{13} &= v_{31} = \frac{1}{\sigma_1}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{1i}-\mu_1)\,\phi^2(u_{1i};\mu_1,\sigma_1)}{\Phi^2(u_{1i};\mu_1,\sigma_1)},\\
v_{14} &= v_{41} = \frac{1}{\sigma_2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{2i}-\mu_2)\,\phi(u_{1i};\mu_1,\sigma_1)\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2)},\\
v_{15} &= v_{51} = -\frac{1}{\theta}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{\phi(u_{1i};\mu_1,\sigma_1)}{\Phi(u_{1i};\mu_1,\sigma_1)},\\
v_{23} &= v_{32} = \frac{1}{\sigma_1}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{1i}-\mu_1)\,\phi(u_{1i};\mu_1,\sigma_1)\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2)},
\end{aligned}$$

$$\begin{aligned}
v_{24} &= v_{42} = \frac{1}{\sigma_2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{2i}-\mu_2)\,\phi^2(u_{2i};\mu_2,\sigma_2)}{\Phi^2(u_{2i};\mu_2,\sigma_2)},\\
v_{25} &= v_{52} = -\frac{1}{\theta}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{2i};\mu_2,\sigma_2)},
\end{aligned}$$

$$
\begin{aligned}
v_{34} &= v_{43} \\
&= \frac{1}{\sigma_1\sigma_2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{1i}-\mu_1)\,(u_{2i}-\mu_2)\,\phi(u_{1i};\mu_1,\sigma_1)\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{1i};\mu_1,\sigma_1)\Phi(u_{2i};\mu_2,\sigma_2)}, \\
v_{35} &= v_{53} = -\frac{1}{\theta\sigma_1}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{1i}-\mu_1)\,\phi(u_{1i};\mu_1,\sigma_1)}{\Phi(u_{1i};\mu_1,\sigma_1)}, \\
v_{45} &= v_{54} = -\frac{1}{\theta\sigma_2}\sum_{i=1}^{n} Var(Z_i|\mathbf{u}_1,\mathbf{u}_2)\frac{(u_{2i}-\mu_2)\,\phi(u_{2i};\mu_2,\sigma_2)}{\Phi(u_{2i};\mu_2,\sigma_2)},
\end{aligned}
$$

and

$$
\begin{aligned}
Var(Z_i|\mathbf{u}_1,\mathbf{u}_2) &= E(Z_i^2|\mathbf{u}_1,\mathbf{u}_2) - E^2(Z_i|\mathbf{u}_1,\mathbf{u}_2) \\
&= \frac{\theta^3 C''''(\theta_*) + 6\theta^2 C'''(\theta_*) + 7\theta_* C''(\theta_*) + C'(\theta_*))}{C'(\theta_*) + \theta_* C''(\theta_*)} \\
&- \left[\frac{\theta^2 C'''(\theta_*) + 3\theta C''(\theta_*) + C'(\theta_*)}{C'(\theta_*) + \theta_* C''(\theta_*)}\right]^2.
\end{aligned}
$$

Applying (7.2) and (7.3), we obtain the observed information as

$$
I(\widehat{\mathbf{\Psi}};\mathbf{u}_1,\mathbf{u}_2) = l_c\left(\widehat{\mathbf{\Psi}};\mathbf{u}_1,\mathbf{u}_2\right) - l_m\left(\widehat{\mathbf{\Psi}};\mathbf{u}_1,\mathbf{u}_2\right).
$$

The standard errors of the MLEs of the EM-algorithm are the square root of the diagonal elements of the $I(\widehat{\mathbf{\Psi}};\mathbf{u}_1,\mathbf{u}_2)$.

8 Simulation

This section provides the results of simulation study. Simulations have been performed in order to investigate the proposed estimators of, $\mu_1, \sigma_1, \mu_2, \sigma_2$ and θ of the proposed EM method. We simulate 1000 times under the BNG distribution with three different sets of parameters and sample sizes n =100, 300 and 500. For each sample size, we compute the using EM algorithm. We also compute the standard error of the EM estimators determined through the Fisher information matrix. The simulated values of $se(\widehat{\mu}_1)$, $se(\widehat{\sigma}_1)$, $se(\widehat{\mu}_2)$, $se(\widehat{\sigma}_2)$, $se(\widehat{\theta})$, $Cov(\widehat{\mu}_1,\widehat{\sigma}_1)$, $Cov(\widehat{\mu}_1,\widehat{\mu}_2)$, $Cov(\widehat{\mu}_1,\widehat{\sigma}_1)$, $Cov(\widehat{\mu}_1,\widehat{\theta})$, $Cov(\widehat{\mu}_2,\widehat{\sigma}_1)$, $Cov(\widehat{\mu}_2,\widehat{\sigma}_2)$, $Cov(\widehat{\mu}_2,\widehat{\theta})$, $Cov(\widehat{\sigma}_1,\widehat{\sigma}_2)$, $Cov(\widehat{\sigma}_1,\widehat{\theta})$ and $Cov(\widehat{\sigma}_2,\widehat{\theta})$, obtained by averaging the corresponding values of the observed information matrices, are computed. All computations are done using R 3.2.1 software. We use the function "nlminb" in Package "stats" for the numerical calculations.

The results for the BNG distribution are shown in Table 1 and Table 2. Some of the points are quite clear from the simulation results: (i) Convergence has been achieved in all cases and this emphasizes the numerical stability of the EM-algorithm. (ii) The differences between the average estimates and the true values are almost small. (iii) These results suggest that the EM estimates have performed consistently. (iv) As the sample size increases, the standard errors of the estimators decrease.

Table 1: Simulated means (AEs) and simulated standard errors (Std) of EM estimators for the BNG distribution.

		Average estimators					Std				
n	$(\mu_1, \sigma_1, \mu_2, \sigma_2, \theta)$	$\widehat{\mu}_1$	$\widehat{\sigma}_1$	$\widehat{\mu}_2$	$\widehat{\sigma}_2$	$\widehat{\theta}$	$\widehat{\mu}_1$	$\widehat{\sigma}_1$	$\widehat{\mu}_2$	$\widehat{\sigma}_2$	$\widehat{\theta}$
100	(0.0,1.0,0.0,1.0,0.1)	-0.0317	0.9896	-0.0334	0.9931	0.1409	0.0150	0.0070	0.0140	0.0070	0.0140
	(0.0,1.0,0.0,1.0,3.0)	0.0214	0.9894	0.0175	0.9949	0.2652	0.0160	0.0070	0.0160	0.0070	0.0160
	(0.0,1.0,0.0,1.0,0.7)	0.1287	0.9794	0.1519	0.9749	0.6113	0.0230	0.0070	0.0220	0.0080	0.0120
200	(0.0,1.0,0.0,1.0,0.1)	-0.0129	0.9956	-0.0158	0.9969	0.1170	0.0073	0.0035	0.0068	0.0036	0.0076
	(0.0,1.0,0.0,1.0,3.0)	0.0203	0.9977	0.0258	0.9904	0.2689	0.0091	0.0036	0.0080	0.0036	0.0086
	(0.0,1.0,0.0,1.0,0.7)	0.1426	0.9791	0.1658	0.9686	0.6139	0.0123	0.0038	0.0109	0.0040	0.0060
500	(0.0,1.0,0.0,1.0,0.1)	-0.0031	0.9973	-0.0058	0.9978	0.1052	0.0030	0.0015	0.0029	0.0014	0.0036
	(0.0,1.0,0.0,1.0,3.0)	0.0094	0.9976	0.0162	0.9977	0.2861	0.0036	0.0015	0.0033	0.0014	0.0036
	(0.0,1.0,0.0,1.0,0.7)	0.1394	0.9795	0.1587	0.9762	0.6216	0.0054	0.0014	0.0049	0.0016	0.0027

Table 2: Simulated covariance between the EM estimators for the BNG distribution.

		Cov									
n	$(\mu_1, \sigma_1, \mu_2, \sigma_2, \theta)$	$(\widehat{\mu}_1, \widehat{\sigma}_1)$	$(\widehat{\mu}_1, \widehat{\mu}_2)$	$(\widehat{\mu}_1, \widehat{\sigma}_2)$	$(\widehat{\mu}_1, \widehat{\theta})$	$(\widehat{\mu}_2, \widehat{\sigma}_1)$	$(\widehat{\mu}_2, \widehat{\sigma}_2)$	$(\widehat{\mu}_2, \widehat{\theta})$	$(\widehat{\sigma}_1, \widehat{\sigma}_2)$	$(\widehat{\sigma}_1, \widehat{\theta})$	$(\widehat{\sigma}_2, \widehat{\theta})$
100	(0.0,1.0,0.0,1.0,0.1)	-0.0016	0.0102	-0.0004	0.0138	-0.0002	-0.0013	0.0134	-0.0001	-0.0007	-0.0004
	(0.0,1.0,0.0,1.0,3.0)	-0.0011	0.0159	-0.0013	0.0194	0.0001	-0.0027	0.0196	0.0004	0.0004	-0.0015
	(0.0,1.0,0.0,1.0,0.7)	-0.0070	0.0442	-0.0096	-0.0225	-0.0027	-0.0131	0.0220	0.0010	-0.0013	-0.0046
200	(0.0,1.0,0.0,1.0,0.1)	0.0001	0.0054	-0.0002	0.0079	0.0002	-0.0004	0.0072	-0.0001	0.0004	0.0001
	(0.0,1.0,0.0,1.0,3.0)	-0.0011	0.0104	-0.0005	0.0126	-0.0003	-0.0011	0.0109	0.0000	-0.0004	-0.0005
	(0.0,1.0,0.0,1.0,0.7)	-0.0044	0.0231	-0.0044	0.0071	-0.0019	-0.0058	0.0113	0.0005	-0.0009	-0.0021
500	(0.0,1.0,0.0,1.0,0.1)	-0.0001	0.0026	-0.0001	0.0041	-0.0001	-0.0001	0.0040	0.0000	-0.0002	0.0000
	(0.0,1.0,0.0,1.0,3.0)	-0.0002	0.0045	-0.0004	0.0053	-0.0001	-0.0007	0.0047	0.0001	0.0000	-0.0004
	(0.0,1.0,0.0,1.0,0.7)	-0.0017	0.0120	-0.0021	0.0067	-0.0008	-0.0026	0.0060	0.0002	-0.0005	-0.0011

Table 3: Kolmogorov-Smirnov statistic and the associated P-value.

Dist.↓	X_1	X_2
NG	0.0637 (0.9956)	0.1466 (0.3275)
NP	0.0864 (0.9123)	0.1462 (0.3309)
NL	0.1177 (0.6052)	0.1467 (0.3265)
N	0.1319 (0.4577)	0.1491 (0.3079)

Table 4: Parameter estimates, AIC and BIC for air pollution data.

Dist.	Parameter estimates	$-\log(L)$	AIC	BIC
BNG	$\widehat{\mu}_1$=-76.90, $\widehat{\sigma}_1$=40.62, $\widehat{\mu}_2$=-35.69, $\widehat{\sigma}_2$=11.99, $\widehat{\theta}$=0.99	305.2414	620.4827	629.1711
BNP	$\widehat{\mu}_1$=56.87, $\widehat{\sigma}_1$=19.38, $\widehat{\mu}_2$=3.06, $\widehat{\sigma}_2$=6.63, $\widehat{\theta}$=3.96	306.4793	622.9587	631.647
BNL	$\widehat{\mu}_1$= 66.78, $\widehat{\sigma}_1$=16.45, $\widehat{\mu}_2$= 6.82, $\widehat{\sigma}_2$= 5.63, $\widehat{\theta}$=0.76	308.7014	627.4028	636.0911
BN	$\widehat{\mu}_1$= 73.85 $\widehat{\sigma}_1$=17.33, $\widehat{\mu}_2$= 9.40, $\widehat{\sigma}_2$= 5.56, $\widehat{\rho}$=0.319	307.8487	625.6973	634.3857
IBN	$\widehat{\mu}_1$= 73.85 $\widehat{\sigma}_1$=17.33, $\widehat{\mu}_2$= 9.40, $\widehat{\sigma}_2$= 5.56	310.104	628.208	635.159

9 Application

In this section, we try to illustrate the better performance of the proposed model. We fit BNG, BNP and BNL models to a real data set. We also fit the bivariate normal (BN) and independent bivariate normal (IBN) distributions to make a comparison with the NPS models. This data, taken from Johnson and Wichern (1992), are related to air pollution. Here, we consider two variables of these data, viz., Solar rad (X_1) and O_3 (X_2). We first test the fitting of marginal distributions. To fit the marginal distributions to this data set, we firstly, compute the KS (Kolomogrov-Smirnov) statistic between the empirical and fitted cumulative distribution functions. The associated P-value (in bracket) of KS statistic for X_1 and X_2 have been given in Table 3. These results suggest that NG, NP, NL and normal distributions, as the marginal distributions of X_1 and X_2, can give a reasonable fit to this data.

For comparison purposes, we estimate parameters by numerically maximizing the likelihood function. The MLEs of the parameters, the maximized log likelihood, the AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) for the BNG, BNP, BNL, BN and IBN models are given in Table 4.

As is well known, a model with a minimum AIC value is to be preferred. Therefore BNG distribution provides a better fit to this data set than the other distributions and hence could be chosen as the best distribution. Now we would like to check whether the BNG distribution fits the bivariate data set or not. For that we have used a copula goodness-of-fit test. Genest et al. (2009) presented a review and comparison on the

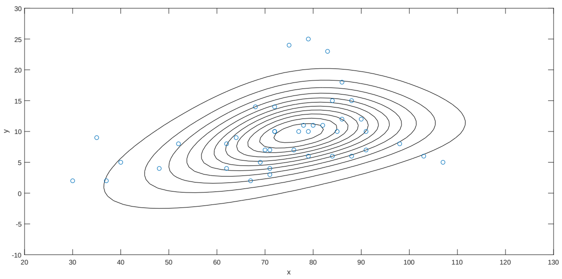

Figure 9: The contour plot of fitted BNG distribution versus air pollution bivariate data

goodness of fit for copulas. One can construct test

$$S_n = \sum_{i=1}^{n} \left(C_n(u_i, v_i) - C_{\widehat{\theta}}(u_i, v_i) \right)^2,$$

where C_n and $C_{\widehat{\theta}}$ are the empirical copula and the fitted copula of the data respectively, with

$$u_i = \frac{\text{rank of } x_i \text{ among } x_1, ..., x_n}{n+1} \quad \text{and} \quad v_i = \frac{\text{rank of } y_i \text{ among } y_1, ..., y_n}{n+1}.$$

The statistic S_n is called Cramer-von Mises statistic. This statistic measures how close the fitted copula is from the empirical copula of data. approximate P-values can be obtained via parametric bootstrap procedure described in Appendix A of Genest et al. (2009). The bootstrap values $S^{*(1)}, ..., S^{*(1000)}$ of the Cramer-von Mises test statistic are generated and approximate P-value is given by $\frac{1}{1000} \sum_{i=1}^{1000} I\left(S^{*(i)} > S_n\right)$. For air pollution date we obtained $S_n = 0.0673$ and P-value$= 0.8021$. Thus we may conclude that BNG distribution performs a good fit to this data set. Figure 9 show that the BNG distribution gives a good fit to the air pollution bivariate data.

10 Conclusion

In this paper we have introduced the bivariate normal-power series class of distributions whose marginals are normal power series distributions. Several statistical properties of this new bivariate distribution have been studied. The estimation of the unknown parameters of the proposed distribution is approached by the EM-algorithm. Finally, we fitted BNPS models to a real data set to show the potential of the new proposed class. Now we briefly discuss a generalization of the proposed model. Let $\{(X_{1n}, X_{2n}); n = 1, 2, ...\}$ be sequence of mutually independent and identically distributed (i.i.d.) bivariate normal random variable with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, where $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and

$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ & \sigma_2^2 \end{pmatrix}$. Suppose N is a power series distribution. Let

$$U_1 = \max(X_{11}, X_{12}, ..., X_{1N}) \quad \text{and} \quad U_2 = \max(X_{21}, X_{22}, ..., X_{2N}).$$

The joint cdf of (U_1, U_2) is

$$F_{U_1,U_2}(u_1, u_2) = \frac{C(\theta\Phi_2(u_1, u_2; \boldsymbol{\mu}, \mathbf{\Sigma}))}{C(\theta)}.$$

for $u_1, u_2 \in \mathbb{R}$. Here $\Phi_2(\cdot\ ; \boldsymbol{\mu}, \mathbf{\Sigma})$ is cdf of bivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$.
We are currently working on this subject and hope to report these findings in a future paper.

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